Modeling with TDD

Part II

Introduction to modeling: the shallow water model

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Motivation

- Last time:
  - TDD
  - modeling example: decay model
- Is TDD a technique that can be applied to modeling projects in the earth science community?
- Is TDD advantageous?

Motivation

- A typical modeling project proceeds in the following steps
  1. decision what to model
  2. mathematical formulation of the model
  3. numerical solution of the model
  4. analysis / understanding of the model output
  5. comparison with data /validation
  6. hypothesis testing / prediction
- Most of the times these steps are iterated several times

Fundamental Conservation Principles

- There are several conservation principles that need to be considered when studying fluid motions. These are
  1. Conservation of momentum (Newton's laws of motion)
  2. Conservation of mass
  3. Conservation of internal energy (heat)
  4. Conservation of salt.
  5. Equation of state $\rho=\rho(S,T,p)$
- All these equations[1] are coupled with each other, which makes the equations describing fluid motions a coupled system of partial differential equations.

[1]Primitive equations

$$p = \rho R_d T$$

$$\frac{d\rho}{dt} = -\rho \nabla . \mathbf{V}$$

$$\frac{du}{dt} = 2\Omega v \sin\theta - 2\Omega w \cos\theta + \frac{uv}{r}\tan\theta - \frac{uw}{r} - \frac{1}{\rho r \cos\theta}\frac{\partial p}{\partial \lambda} + \tau_\lambda$$

$$\frac{dv}{dt} = -2\Omega u \sin\theta - \frac{u^2}{r}\tan\theta - \frac{vw}{r} - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \tau_\theta$$

$$\frac{dw}{dt} = 2\Omega u \cos\theta + \frac{u^2 + v^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} - g + \tau_r$$

$$c_{p_d}\frac{dT}{dt} = Q + \frac{R_d T}{p}\frac{dp}{dt}$$

“dynamical core” equations

- Shallow water equations
- Isopycnic/isentropic equations
- Compressible Euler equations
- Incompressible Euler equations
- Boussinesq-type approximations
- Anelastic equations
- Primitive equations
- Pressure or mass coordinate equations

Decide what to model

- Classical (and educational) approach: Simplify the governing equations *BEFORE* numerics is introduced (e.g. by scale analysis, hydrostatic approximation, Boussinesq or Anelastic approximation) depending on the context.
- Solutions to the governing equations have three important propagation speeds:
  - acoustic waves (speed of sound),
  - gravity waves (gravity-wave speed), and
  - advective motion (wind speed), which affect the time-step that can be used in numerical procedures (constraint : cdt<dx).

Scale analysis

Example :

Typical observed values for mid-latitude synoptic systems:

U ~ 10 ms^{-1}

W ~ 10^{-2} ms^{-1}

L ~ 10^6 m

$\Delta p / \tilde{\rho}$ ~ 10^3 m^2s^{-2}

f_0 ~ 10^{-4} s^{-1} $(\mathrm{f} = 2\Omega \sin\theta)$

a ~ 10^7 m

H ~ 10^4 m

Mathematical formulation of the model

Starting from the primitive equations, assume:

1. Homogeneous fluid : ρ=const
2. a shallow atmosphere with radius $r = a + z \sim a$
3. Allow to drop Coriolis and metric terms which depend on *w*
4. Shallow water approximation : $D/L \ll 1$ where D and L are the characteristic scales of the motion in the vertical and horizontal respectively
   - leads to the so called hydrostatic approximation.

Mathematical formulation of the model

$$\frac{\partial u}{\partial t} = -fv - g\frac{\partial \eta}{\partial x} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y}$$

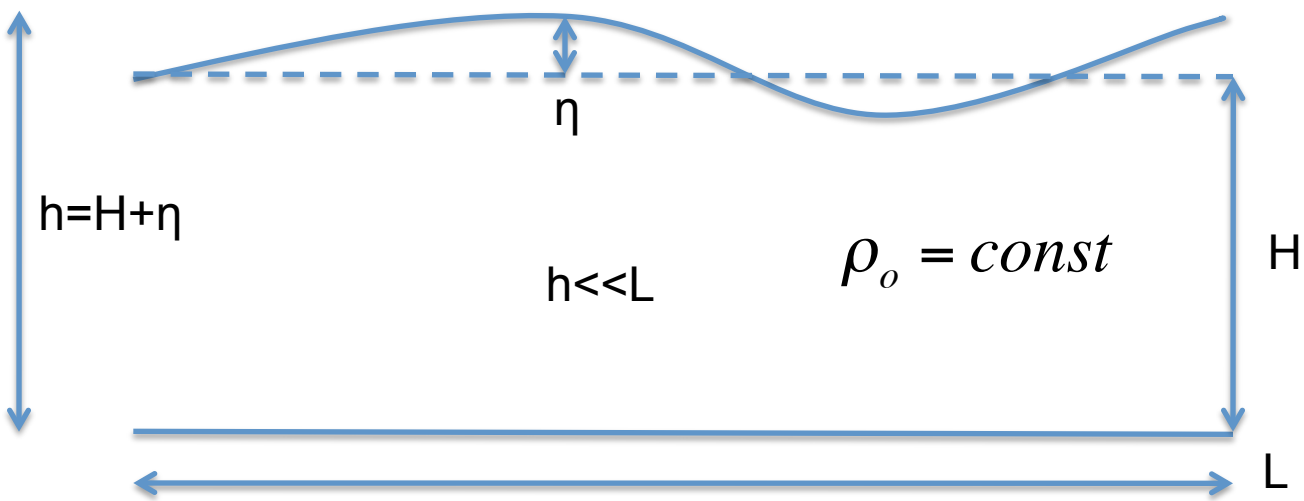
$$\frac{\partial v}{\partial t} = +fu - g\frac{\partial \eta}{\partial y} - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y}$$

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}[hu] - \frac{\partial}{\partial y}[hv]$$

$$where$$

Reference: Gill (1982)

$$h = H(x,y) + \eta(x,y,t)$$



Mathematical formulation of the model

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{u^2 + v^2}{2}\right) - v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u^2 + v^2}{2}\right) + u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

These allow momentum equations to be written in an alternate form (conservative form) making reference to KE,PE and vorticity

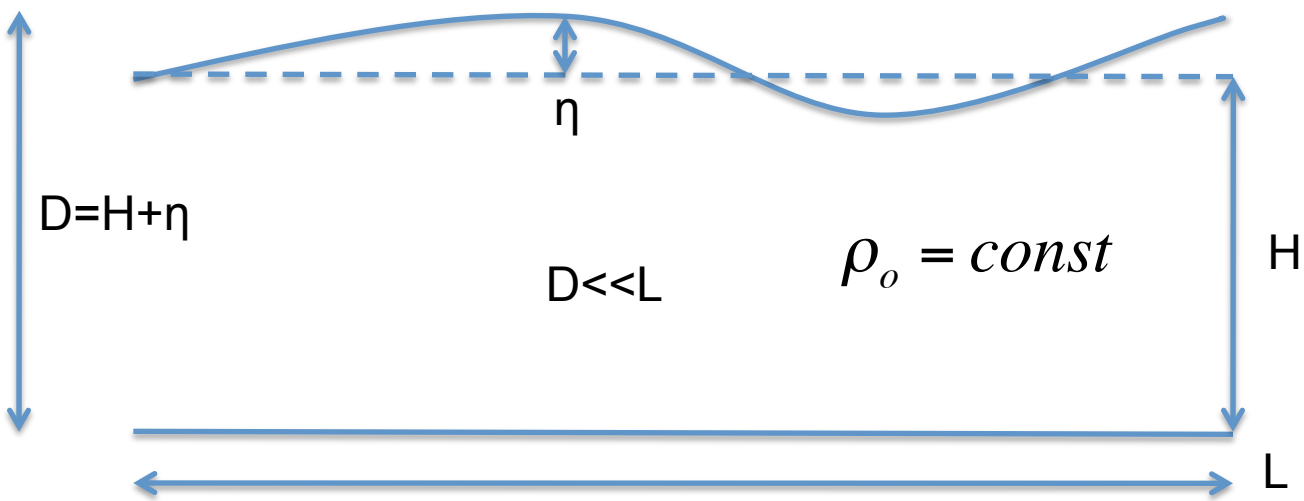
Reference: Gill (1982)

Mathematical formulation of the model

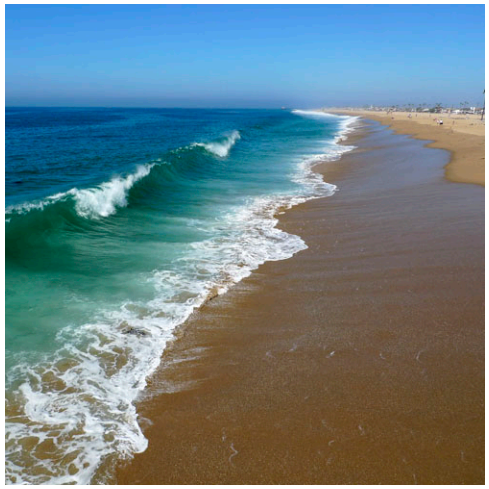
$$\frac{\partial u(H+\eta)}{\partial t} = -fv(H+\eta) - g(H+\eta)\frac{\partial \eta}{\partial x} - \frac{\partial u^2(H+\eta)}{\partial x} - \frac{\partial uv(H+\eta)}{\partial y}$$

$$\frac{\partial v(H+\eta)}{\partial t} = fu(H+\eta) - g(H+\eta)\frac{\partial \eta}{\partial y} - \frac{\partial uv(H+\eta)}{\partial x} - \frac{\partial v^2(H+\eta)}{\partial y}$$

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\left[(H+\eta)u\right] - \frac{\partial}{\partial y}\left[(H+\eta)v\right]$$

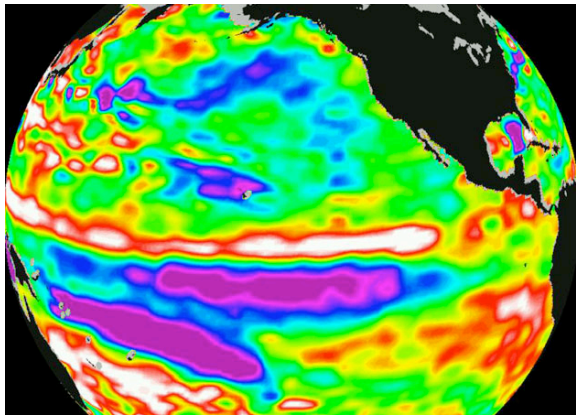


Applications of the shallow water model

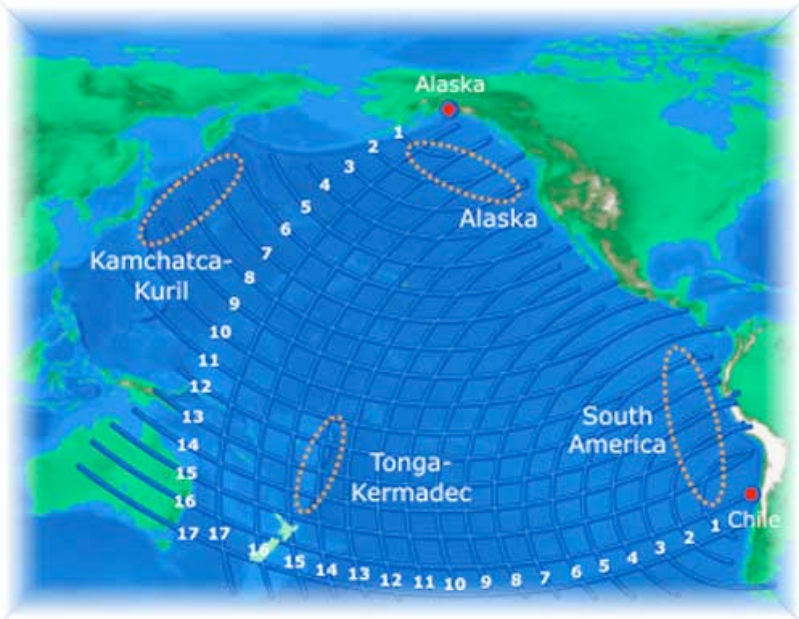


Long waves

El Nino



Tsunamis

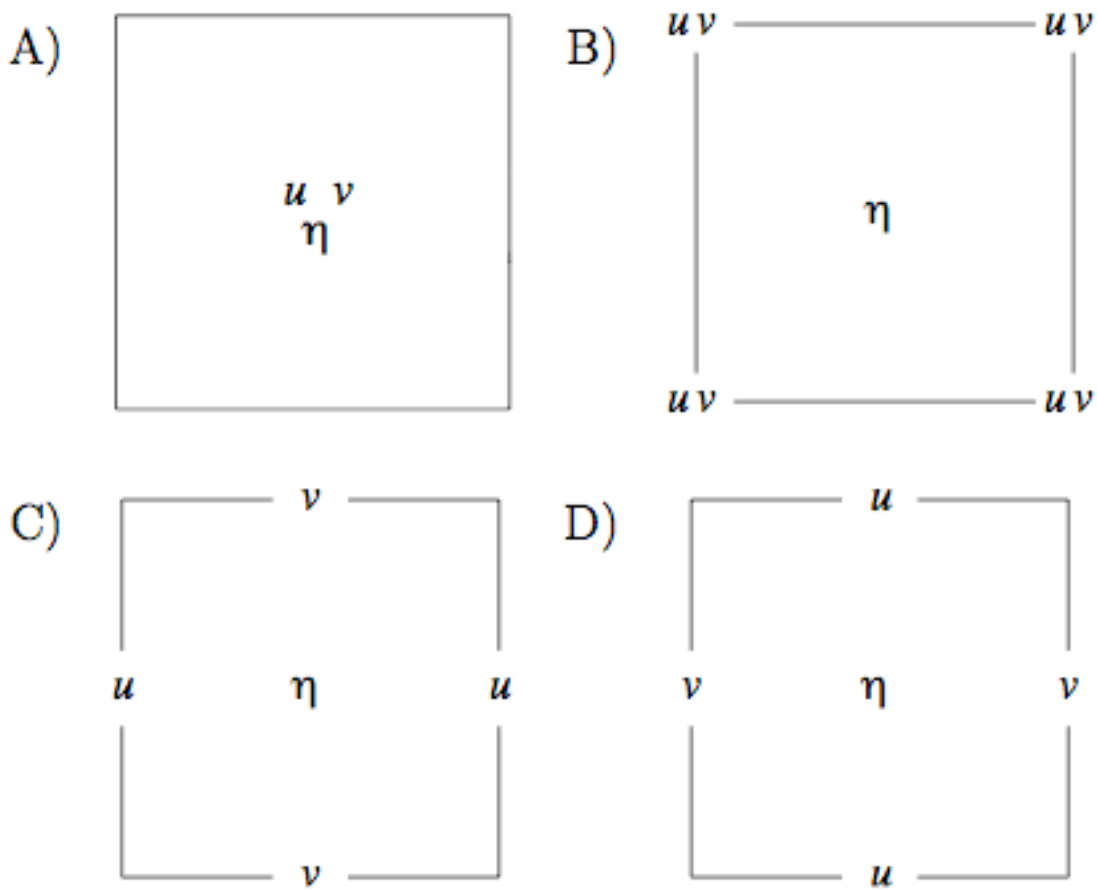


Waves in bathtub

Numerical solution of the model

- Rectangular basin – solid boundaries (no normal flow)
- Regular grid in space, i.e. the water is subdivided into 2-D boxes (for simplicity, all boxes have the same width and length Δx), numbered by i for the x-direction and by j for the y-direction. There are in total nx × ny boxes.

Arrangements of shallow water variables



Finite differencing

- A grid:

$$
\begin{aligned}
\partial_t u - fv + \frac{g}{\Delta x}\delta_i \overline{\eta}^i &= 0 \\
\partial_t v + fu + \frac{g}{\Delta y}\delta_j \overline{\eta}^j &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^i + \frac{H}{\Delta y}\delta_j \overline{v}^j &= 0
\end{aligned}
$$

- B grid:

$$
\begin{aligned}
\partial_t u - fv + \frac{g}{\Delta x}\delta_i \overline{\eta}^j &= 0 \\
\partial_t v + fu + \frac{g}{\Delta y}\delta_j \overline{\eta}^i &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^j + \frac{H}{\Delta y}\delta_j \overline{v}^i &= 0
\end{aligned}
$$

- C grid:

$$
\begin{aligned}
\partial_t u - f\overline{v}^{ij} + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f\overline{u}^{ij} + \frac{g}{\Delta y}\delta_j \eta &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u + \frac{H}{\Delta y}\delta_j v &= 0
\end{aligned}
$$

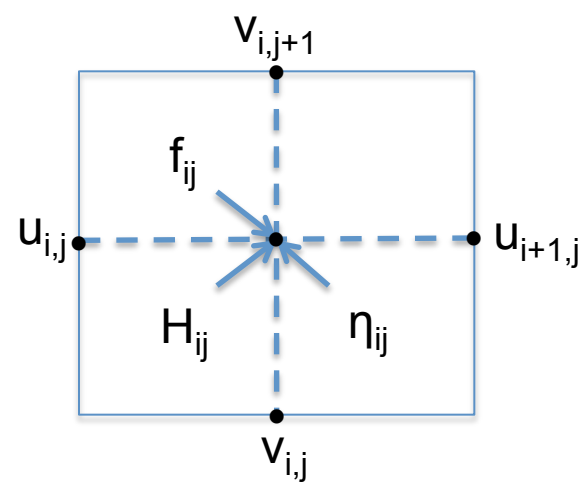
- D grid:

$$
\begin{aligned}
\partial_t u - f\overline{v}^{ij} + \frac{g}{\Delta x}\delta_i \overline{\eta}^{ij} &= 0 \\
\partial_t v + f\overline{u}^{ij} + \frac{g}{\Delta y}\delta_j \overline{\eta}^{ij} &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^{ij} + \frac{H}{\Delta y}\delta_j \overline{v}^{ij} &= 0
\end{aligned}
$$

Numerical solution of the model

- Solution is more stable if we use a staggered grid – choose the C-grid.

$$\frac{\partial u(H+\eta)}{\partial t} = -fv(H+\eta) - g(H+\eta)\frac{\partial \eta}{\partial x} - \frac{\partial u^2(H+\eta)}{\partial x} - \frac{\partial uv(H+\eta)}{\partial y}$$



- Time stepping: leapfrog scheme
  - 3 time levels

$$x^{n+1} = x^{n-1} + 2\Delta t RHS^n$$

Numerical solution of the model

$$\frac{\partial u(H+\eta)}{\partial t} = -fv(H+\eta) - g(H+\eta)\frac{\partial \eta}{\partial x} - \frac{\partial u^2(H+\eta)}{\partial x} - \frac{\partial uv(H+\eta)}{\partial y}$$

$$\frac{\partial v(H+\eta)}{\partial t} = +fu(H+\eta) - g(H+\eta)\frac{\partial \eta}{\partial y} - \frac{\partial uv(H+\eta)}{\partial x} - \frac{\partial v^2(H+\eta)}{\partial y}$$

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\left[(H+\eta)u\right] - \frac{\partial}{\partial y}\left[(H+\eta)v\right]$$

Coriolis

pressure

nonlinear terms

finite differencing

time stepping

Numerical solution of the model – enter TDD

- What do we test first?
- Indentified various computations that need to be performed
- Start with a test that calculates pressure gradients terms

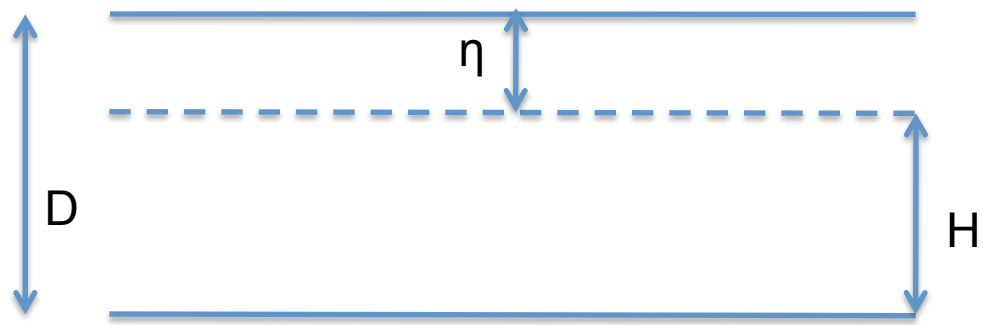
$$g(H+\eta)\frac{\partial \eta}{\partial x}$$

  - Note that on C-grid H and eta are defined on the center of the box
  - C-grid requires that u,v dimensions be u(nx+1,ny) and v(nx,ny+1) respectively. Also, we assume the boundary condition that all velocities vanish on the boundaries.

Numerical solution of the model – tests

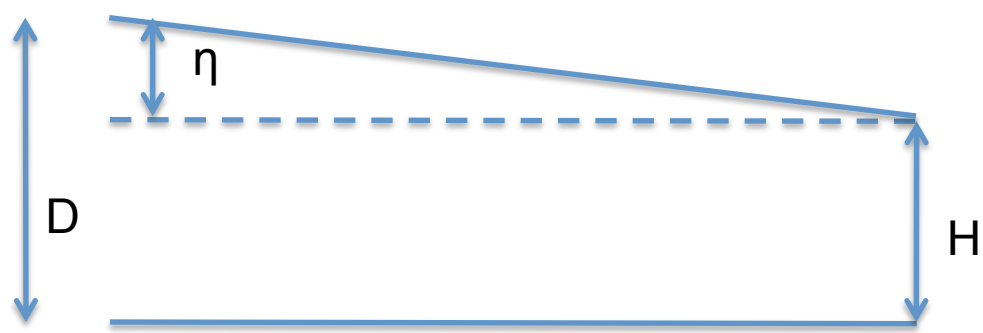
$$g(H+\eta)\frac{\partial \eta}{\partial x}$$

- If η=const then PGF=0



- If η=x then PGF(x)=g(H+η), PGF(x)=0
- If η=y then PGX(x)=0, PGF(y)=g(H+η)

- nx=3, ny=3
- dx = dy = 1



Some observations

- TDD can be be used to drive the model development of a shallow water model....but
  - We now we need a testing framework: pFUnit is adequate
  - Need to bootstrap setup: start testing immediately, otherwise the additional "overhead" of getting spun up may be discouraging to modelers
  - It can be difficult (and time consuming) to come up with meaningful tests
  - Discipline is required
- Is it advantageous to use TDD to develop earth system models?
  - TDD can blend in within the flow of modeling project
  - Can be used as a tool to teach modeling
  - Keep in mind advantages of TDD: e.g. extensibility

The End